HOMEWORK 8 – ANSWERS TO (MOST) PROBLEMS

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Section 4.2: The Mean Value Theorem

4.2.6. $f(0) = f(\pi) = 0$, but $f'(x) = \sec^2(x) > 0$. This does not contradict Rolle's Theorem because f is not continuous on $[0, \pi]$ (it is discontinuous at $\frac{\pi}{2}$.

4.2.11. $\ln(x)$ is continuous on $[1, 4]$, differentiable on $(1, 4)$; $c = \frac{3}{\ln(4)}$

4.2.18. Let $f(x) = x^3 + e^x$

At least one root: $f(-1) = -1 + e^{-1} < 0$ and $f(0) = 0 + e^0 = 1 > 0$ and f is continuous, so by the **Intermediate Value Theorem (IVT)** the equation has at least one root.

At most one root: Suppose there are two roots a and b. Then $f(a) = f(b) = 0$, so by **Rolle's Theorem** there is at least one $c \in (a, b)$ such that $f'(c) = 0$. But $f'(c) = 3c^2 + e^c \geq e^c > 0$, so $f'(c) \neq 0$, which is a contradiction, and hence the equation has at most one root.

4.2.23. By the MVT, $\frac{f(4)-f(1)}{4-1} = f'(c)$ for some c in $(1, 4)$. Solving for $f(4)$ and using $f(1) = 10$, we get $f(4) = 3f'(c) + 10 \geq 6 + 10 = 16$.

4.2.29. This is equivalent to showing:

$$\left| \frac{\sin(a) - \sin(b)}{a - b} \right| \leq 1$$

Which is the same as:

$$\left| \frac{\sin(b) - \sin(a)}{b - a} \right| \leq 1$$

Which is the same as:

$$-1 \leq \frac{\sin(b) - \sin(a)}{b - a} \leq 1$$

But by the MVT applied to $f(x) = \sin(x)$, we get:

$$\frac{\sin(b) - \sin(a)}{b - a} = \cos(c)$$

for some c in (a, b). However, $-1 \leq \cos(c) \leq 1$, and so we're done!

Date: Friday, November 1st, 2013.

4.2.32. Let $f(x) = 2\sin^{-1}(x)$, $g(x) = \cos^{-1}(1-2x^2)$.

Then $f'(x) = \frac{2}{\sqrt{1-x^2}}$ and:

$$g'(x) = -\frac{-4x}{\sqrt{1-(1-2x^2)^2}} = \frac{4x}{\sqrt{1-1+4x^2-4x^4}} = \frac{4x}{\sqrt{4x^2-4x^4}} = \frac{4x}{2x\sqrt{1-x^2}} = \frac{2}{\sqrt{1-x^2}} = f'(x)$$

(you get this by factoring out $4x^2$ out of the square root. This works because $x \geq 0$)

Hence $f'(x) = g'(x)$, so $f(x) = g(x) + C$

To get C, plug in $x = 0$, so $f(0) = g(0) + C$. But $f(0) = g(0) = 0$, so $C = 0$, whence $\boxed{f(x) = g(x)}$

4.2.34. Let $f(t)$ be the speed at time t. By the MVT with $a = 2:00$ and $b = 2:10$, we get:

$$\frac{f(2:10) - f(2:00)}{2:10 - 2:00} = f'(c)$$

But $2:10 - 2:00 = 10$ minutes $= \frac{1}{6}$ h, so:

$$\frac{50-30}{\frac{1}{6}} = f'(c)$$

Whence: $\boxed{f'(c) = 120}$ for some c between 2 : 00 pm and 2 : 10 pm. But $f'(c)$ is the acceleration at time c, and so we're done!

Section 4.3: How derivatives affect the shape of a graph

4.3.2.

(a) $(0,1) \cup (3,7)$
(b) $(1,3)$
(c) $(2,4) \cup (5,7)$
(d) $(0,2) \cup (4,5)$
(e) $(2,2)$, $(4,2.5)$, $(5,4)$

4.3.9.

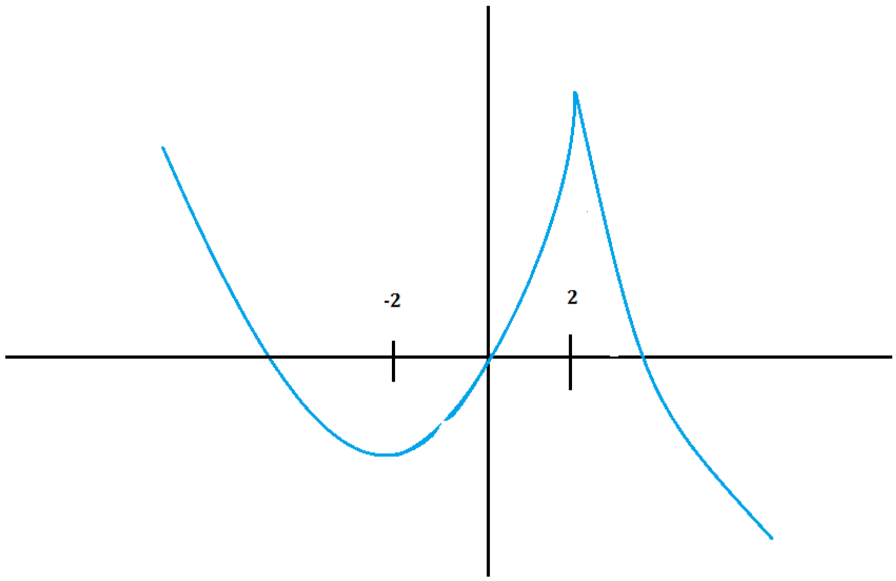
(a) $f'(x) = 6x^2 + 6x - 36 = 6(x-2)(x+3)$; $\nearrow$ on $(-\infty,-3) \cup (2,\infty)$, $\searrow$ on $(-3,2)$
(b) Local max: $f(-3) = 81$; Local min: $f(2) = -44$
(c) $f''(x) = 12x+6$; CU on $(-\frac{1}{2},\infty)$, CD on $(-\infty, \frac{-1}{2})$, IP $(-\frac{1}{2}, f(-0.5) = \frac{37}{2})$

4.3.13.

(a) $f'(x) = \cos(x) - \sin(x)$; $\nearrow$ on $(0,\frac{\pi}{4}) \cup (\frac{5\pi}{4},\infty)$, $\searrow$ on $(\frac{\pi}{4},\frac{5\pi}{4})$
(b) Local max: $f(\frac{\pi}{4}) = \sqrt{2}$; Local min: $f(\frac{5\pi}{4}) = -\sqrt{2}$
(c) $f''(x) = -\sin(x) + \cos(x)$; CU on $(\frac{3\pi}{4},\frac{7\pi}{4})$, CD on $(0,\frac{3\pi}{4}) \cup (\frac{7\pi}{4},2\pi)$, IP $(\frac{3\pi}{4},0)$, $(\frac{7\pi}{4},0)$

4.3.27. A possible graph looks like this:

1A/Math 1A - Fall 2013/Solution Bank/Concave-Kink.png



4.3.33.

(a) $f'(x) = 3x^2 - 12$, $\nearrow$ on $(\infty, -2) \cup (2, \infty)$, $\searrow$ on $(-2, 2)$
(b) Local min: $f(2) = -14$, Local max: $f(-2) = 18$
(c) $f''(x) = 6x$; CD on $(-\infty, 0)$, CU on $(0, \infty)$; Inflection point at $(0, 2)$
(d) Draw the graph!

4.3.43.

(a) $f'(\theta) = -2\sin(\theta) - 2\cos(\theta)\sin(\theta) = -2\sin(\theta)(1 + \cos(\theta))$; $\nearrow$ on $(\pi, 2\pi)$, $\searrow$ on $(0, \pi)$
(b) Local min: $f(\pi) = -1$, no local max.
(c) $f''(x) = -2\cos(\theta) + 2\sin^2(\theta) - 2\cos^2(\theta) = -2\cos(\theta) + 2 - 4\cos^2(\theta) = -4(\cos^2(\theta) - \frac{\cos(\theta)}{2} - \frac{1}{2}) = -4(\cos(\theta) + 1)(\cos(\theta) - \frac{1}{2})$; CU on $(\frac{\pi}{3}, \frac{5\pi}{3})$, CD on $(0, \frac{\pi}{3}) \cup (\frac{5\pi}{3}, 2\pi)$, IP $(\frac{\pi}{3}, \frac{5}{4}), (\frac{5\pi}{3}, \frac{5}{4})$
(d) Draw the graph!

4.3.45.

(a) VA: $x = 0$, HA: $y = 1$ (at $\pm\infty$)
(b) $f'(x) = -\frac{1}{x^2} + \frac{2}{x^3} = \frac{2-x}{x^3}$; $\searrow$ on $(\infty, 0) \cup (2, \infty)$, $\nearrow$ on $(0, 2)$

(c) Local maximum at $(2, \frac{5}{4})$, No local minimum
(d) $f''(x) = \frac{-6x^2+2x^3}{x^6} = \frac{-6+2x}{x^4} = \frac{2x-6}{x^4}$; CD on $(-\infty, 0) \cup (0, 3)$, CU on $(3, \infty)$; IP at $(3, \frac{11}{9})$
(e) Draw the graph!

4.3.49.

(a) No VA; HA: $y = 0$ (at $\pm\infty$)
(b) $f'(x) = (-2x)e^{-x^2}$, $\nearrow$ on $(-\infty, 0)$, $\searrow$ on $(0, \infty)$
(c) Local maximum at $(0, 1)$, no local minimum
(d) $f''(x) = (-2+4x^2)e^{-x^2} = 2(2x^2-1)e^{-x^2}$; CU on $\left(-\infty, -\frac{1}{\sqrt{2}}\right) \cup \left(\frac{1}{\sqrt{2}}, \infty\right)$; IP at $\left(\pm\frac{1}{\sqrt{2}}, e^{-\frac{1}{2}}\right)$
(e) Draw the graph!

4.3.77. Let $f(x) = \tan(x) - x$, then $f'(x) = \sec^2(x) - 1 = 1 + \tan^2(x) + 1 - 1 = \tan^2(x) > 0$ on $\left(0, \frac{\pi}{2}\right)$, hence f is increasing on $\left(0, \frac{\pi}{2}\right)$. In particular, $f(x) > f(0) = 0$, so $\tan(x) - x > 0$, so $\tan(x) > x$ on $\left(0, \frac{\pi}{2}\right)$

Section 4.4: L'Hopital's Rule

4.4.3.

(a) No, $-\infty$
(b) Yes, $\infty - \infty$
(c) No, ∞

4.4.4.

(a) Yes, 0^0
(b) No, 0
(c) Yes, 1^∞
(d) Yes, ∞^0
(e) No, ∞
(f) Yes, ∞^0

4.4.11.

$$\lim_{x\to\left(\frac{\pi}{2}\right)^+} \frac{\cos(x)}{1-\sin(x)} = \lim_{x\to\left(\frac{\pi}{2}\right)^+} \frac{-\sin(x)}{-\cos(x)} = \frac{-1}{-0^-} = \frac{-1}{0^+} = -\infty$$

4.4.17.

$$\lim_{x\to\infty} \frac{\ln(x)}{\sqrt{x}} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0$$

4.4.29.

$$\lim_{x\to 0} \frac{\sin^{-1}(x)}{x} = \lim_{x\to 0} \frac{\frac{1}{\sqrt{1-x^2}}}{1} = 1$$

4.4.45.

$$\lim_{x\to\infty} x^3 e^{-x^2} = \lim_{x\to\infty} \frac{x^3}{e^{x^2}} = \lim_{x\to\infty} \frac{3x^2}{2xe^{x^2}} = \frac{3}{2}\lim_{x\to\infty} \frac{x}{e^{x^2}} = \frac{3}{2}\lim_{x\to\infty} \frac{1}{2xe^{x^2}} = \frac{3}{2} \times 0 = 0$$

4.4.58.

1) Let $y = \left(1 + \frac{a}{x}\right)^{bx}$
2) $\ln(y) = bx \ln(1 + \frac{a}{x})$
3)

$$\lim_{x\to\infty} \ln(y) = \lim_{x\to\infty} bx\ln(1+\frac{a}{x}) = \lim_{x\to\infty} \frac{\ln(1+\frac{a}{x})}{\frac{1}{bx}} = \lim_{x\to\infty} \frac{(\frac{1}{1+\frac{a}{x}})(-\frac{a}{x^2})}{(-\frac{1}{x^2})(\frac{1}{b})} = \lim_{x\to\infty} \frac{ab}{1+\frac{a}{x}} = ab$$

4) So $\lim_{x\to\infty} \left(1 + \frac{a}{x}\right)^{bx} = e^{ab}$

4.4.61.

1) Let $y = x^{\frac{1}{x}}$
2) Then $\ln(y) = \frac{\ln(x)}{x}$
3) So $\lim_{x\to\infty} \ln(y) = \lim_{x\to\infty} \frac{\ln(x)}{x} = 0$
4) Hence $\lim_{x\to\infty} x^{\frac{1}{x}} = \lim_{x\to\infty} y = e^0 = 1$

4.4.77. All you gotta do is show that:

$$\lim_{n\to\infty} \left(1 + \frac{r}{n}\right)^{nt} = e^{rt}$$

1) Let $y = \left(1 + \frac{r}{n}\right)^{nt}$
2) $\ln(y) = nt \ln(1 + \frac{r}{n})$
3) The important thing to realize is that you're taking the limit as n goes to ∞, which means that r and t are **constants**!

$$\lim_{n\to\infty} \ln(y) = \lim_{n\to\infty} nt\ln(1+\frac{r}{n}) = \lim_{n\to\infty} \frac{\ln(1+\frac{r}{n})}{\frac{1}{nt}} = \lim_{n\to\infty} \frac{\frac{\frac{-r}{n^2}}{1+\frac{r}{n}}}{-\frac{1}{n^2 t}} = \lim_{n\to\infty} \frac{\frac{rn^2t}{n^2}}{1+\frac{r}{n}} = \lim_{n\to\infty} \frac{rt}{1+\frac{r}{n}} = \frac{rt}{1+0} = rt$$

4) So $\lim_{n\to\infty} \left(1 + \frac{r}{n}\right)^{nt} = e^{rt}$, and hence $\lim_{n\to\infty} A_0 \left(1 + \frac{r}{n}\right)^{nt} = A_0 e^{rt}$